Resolution of the mystery behind Chandrasekhar's black hole transformations

1977 J. Phys. A: Math. Gen. 10 885

(http://iopscience.iop.org/0305-4470/10/6/011)

J. Phys. A: Math. Gen., Vol. 10, No. 6, 1977. Printed in Great Britain.

Resolution of the mystery behind Chandrasekhar's black hole transformations

J Heading

Department of Applied Mathematics, The University College of Wales, Aberystwyth, Dyfed, UK

Received 13 January 1977

Abstract. Investigating the three differential equations in normal form of (i) Zerilli, (ii) Bardeen and Press, (iii) Regge and Wheeler, governing the perturbations of the Schwarzschild black hole, Chandrasekhar has demonstrated the somewhat complicated transformations between these equations. This complication hides the basic nature of the transformations and their mutual connections. The whole scheme can be parametrized, with one condition imposed, yielding for every functional parameter inevitably three potentials of the above types. Any Schrödinger equation in normal form can be similarly treated, but the analogous Bardeen and Press potential is more complicated than the original. Thus an investigation is undertaken as to why the Bardeen and Press potential for the black hole is analytically 'simple'; conditions for this simplicity inevitably lead to this particular potential, and hence to the other two potentials. Every symbol occurring in these three potentials is thereby explained analytically.

1. The transformation problem

Several differential equations of the second order have been produced that govern the perturbations of the Schwarzschild black hole; we may mention those of Zerilli (1970), Bardeen and Press (1973), and Regge and Wheeler (1957); these three equations are all expressible in normal form, though the second contains a potential that is not independent of the wavenumber. Since they are all derived from the same basic physical model, it follows that there must be analytical transformations between these various equations. Chandrasekhar (1975) has produced these transformations in a comprehensive paper in which the Zerilli equation is first derived from the Schwarzschild metric. He commences his paper by stating that 'there continues to be some elements of mystery shrouding the subject', and he sets himself the task of 'resolving some of the unanswered questions'. In the appendix, the same basic ideas and *very similar relations* yield the Regge and Wheeler equation. Exactly why two very similar transformations of the Zerilli equation and the Regge and Wheeler equation yield the identical third equation of Bardeen and Press remains unanswered from a mathematical point of view.

The reason why this investigation presents problems from an analytical point of view is because only the explicit complicated algebraic functions pertaining to the Zerilli equation are used. A deeper insight into the analysis is gained when the equations are considered generally rather than particularly. We therefore show how all the equations involved may be parametrized, yielding a complete class of such equations in triplets. Every analogous Zerilli potential will yield two further equations automatically, corresponding to those of Bardeen and Press, and Regge and Wheeler. At the same time, we allow the independent variable of the differential equations to be distinct from the independent variable of the potentials involved (as in Chandrasekhar's investigations). The apparent algebraic complication of Chandrasekhar's scheme is immediately dissipated when it is seen how the process forms part of a general parametrized transformation containing both arbitrary numerical and functional parameters to generate it.

We then apply the same basic transformation to various examples of Schrödinger's equation; in each case, the corresponding Regge and Wheeler equation merely reproduces the original Schrödinger (Zerilli) equation with different eigenvalues. But the corresponding Bardeen and Press equation is always more complicated. The last section of the paper is therefore occupied with the nature of the simplicity of the Bardeen and Press equation in the black hole analysis, and it is found possible to derive this potential analytically merely from a postulate of simplicity; the remaining two potentials then follow automatically.

For many theorems and analytical results behind such transformations of second-order differential equations in normal form, the reader is referred to a complementary paper by the author (Heading 1977), though the present paper is independent of any consultation of this other paper.

2. The basic equations

The Zerilli equation

$$Z''+QZ=0,$$

its dual, the Regge and Wheeler equation

$$X''+Q_0X=0,$$

and the Bardeen and Press equation

$$\phi''+q\phi=0,$$

where a prime denotes differentiation with respect to z, and

$$Q=k^2-V\equiv k^2-\frac{[2n^2(n+1)r^3+6n^2mr^2+18nm^2r+18m^3](r-2m)}{r^4(nr+3m)^2},$$

$$Q_0=k^2-V_0\equiv k^2-\frac{2[(n+1)r-3m](r-2m)}{r^4},$$

$$q=k^2-\frac{4\mathrm{i}k(r-3m)}{r^2}-\frac{l(l+1)(r-2m)+2m}{r^3},$$

with

$$\frac{\mathrm{d}}{\mathrm{d}z}=\frac{r-2m}{r}\frac{\mathrm{d}}{\mathrm{d}r},$$

and

$$n=\tfrac{1}{2}(l-1)(l+2),$$

are related by the transformations

$$\phi = \alpha Z + \beta Z', \qquad \phi = \alpha_0 X + \beta_0 X',$$

where

$$\alpha = \frac{r^3}{r-2m}(V + \mathrm{i}kW - 2k^2), \qquad \beta = \frac{r^3}{r-2m}(W + 2\mathrm{i}k),$$

$$\alpha_0 = \frac{r^3}{r-2m}(V_0 + \mathrm{i}kW_0 - 2k^2), \qquad \beta_0 = \frac{r^3}{r-2m}(W_0 + 2\mathrm{i}k),$$

$$W = \frac{2(nr^2 - 3nmr - 3m^2)}{r^2(nr+3m)}, \qquad W_0 = \frac{2(r-3m)}{r^2}.$$

Although the two functions W and W_0 are quite different, there is a far greater connection between them than meets the eye at first sight. On account of the partial fractions involved, these transformations are almost too complicated to check directly, so we intend to develop a step-by-step general transformation theory together with a particular class of transformations that yield pairs of dual equations by means of numerical and functional parameters. The above quoted transformations form a particular pair of dual members of the class.

3. The general transformation

Generally, we consider the transformation between the equations

$$Z'' + Q(z)Z = 0, \qquad \phi'' + q(z)\phi = 0$$

by means of the substitution

$$\phi = \alpha Z + \beta Z'. \tag{1}$$

All functions considered are differentiable as many times as necessary in the development of the theory. Then

$$\phi' = \alpha Z' + \alpha' Z + \beta' Z' + \beta Z'' = (\alpha' - \beta Q)Z + (\alpha + \beta')Z', \tag{2}$$

$$\begin{aligned}\phi'' &= (\alpha' - \beta Q)Z' + (\alpha' - \beta Q)'Z + (\alpha + \beta')'Z' - (\alpha + \beta')QZ \\ &= -q(\alpha Z + \beta Z').\end{aligned}$$

Equating coefficients of Z and Z', we have

$$(\alpha' - \beta Q)' - (\alpha + \beta')Q + \alpha q = 0, \tag{3}$$

$$(\alpha' - \beta Q) + (\alpha + \beta')' + \beta q = 0, \tag{4}$$

forming in effect two simultaneous equations for α and β when Q, q are given. Hence

$$Q - q = \frac{\alpha'' - \beta Q' - 2\beta' Q}{\alpha} = \frac{2\alpha' + \beta''}{\beta}. \tag{5}$$

In order that our general scheme should match the particular transformations given by Chandrasekhar, we introduce two new functions y and W defined in terms of α and β by the equations

$$\alpha = y(V + \mathrm{i}kW - 2k^2), \qquad \beta = y(W + 2\mathrm{i}k),$$

where Q is to have the form $k^2 - V$. In other words, y and W are given by

$$y = \frac{\alpha - ik\beta}{V}, \qquad W = \frac{\beta V - 2ik\alpha - 2k^2\beta}{\alpha - ik\beta}.$$

Again, to induce parametrization, introduce functions $\gamma a(z)$ and $b(z)$, γ being a constant which need have only the values ± 1,

$$V' + WV = -\gamma a(z), \tag{6}$$

$$W' + V = b(z). \tag{7}$$

These combinations occurred coincidentally in Chandrasekhar's investigations, but here they appear as a vital part of our parametrization theory.

To remove all reference to k occurring in the denominator of $Q - q$ given by (5), we multiply numerator and denominator of the second ratio by ik, and form the new ratio

$$Q - q = \frac{\alpha'' - \beta Q' - 2\beta' Q - ik(2\alpha' + \beta'')}{\alpha - ik\beta}$$

$$= \frac{\alpha'' - \beta Q' - 2\beta' Q - 2ik\alpha' - ik\beta''}{yV}. \tag{8}$$

Now from (6) and (7),

$$V'' = -(b - V)V - W(-WV - \gamma a) - \gamma a',$$

$$W'' = b' + WV + \gamma a.$$

Consequently we substitute the forms of α, β, V', W', V'', W'' into (8), obtaining after simplification,

$$q = k^2 - \frac{2iky'}{y} - \frac{y''}{y} - b + \frac{\gamma(ay^2)'}{y^2 V}, \tag{9}$$

thereby forming the function that appears in the Bardeen and Press equation.

Returning to the transformation equations (3) and (4), we eliminate q, obtaining

$$\beta[\alpha'' - (\beta Q)' - \alpha Q - \beta' Q] = \alpha(2\alpha' - \beta Q + \beta''),$$

or

$$\alpha\beta'' - \alpha''\beta + 2\alpha\alpha' + (2\beta\beta' Q + \beta^2 Q') = 0,$$

integrating to

$$Q\beta^2 + \alpha^2 + \alpha\beta' - \alpha'\beta = \text{constant}. \tag{10}$$

Upon substitution of α and β, using V' and W', we obtain

$$y^2(bV + \gamma a W + 2ik\gamma a) = \text{constant}. \tag{11}$$

We now differentiate this invariant form, substituting the values of V' and W' given by (6) and (7):

$$(y^2 b)'V + (y^2 b)(-WV - \gamma a) + \gamma(y^2 a)'W + \gamma y^2 a(b - V) + 2ik\gamma(y^2 a)' = 0,$$

which may be re-arranged to

$$V[(y^2b)' - W(y^2b) - \gamma y^2 a] + (y^2a)'\gamma(W + 2ik) = 0. \tag{12}$$

It is this result that leads to the parametrization of the whole scheme.

4. Parametrization of the Chandrasekhar scheme

Thus far, equation (12) is completely general, but its structure appears to prevent parametrization of the general scheme. At this stage, therefore, we must specialize the class of equations under consideration to be that for which the Zerelli equation is a particular case.

We restrict ourselves to the condition

$$y^2 a = M,$$

a constant. The attentive reader will note that this condition is implied in the algebra of the Chandrasekhar scheme, and we would stress that this is the *only* condition that we are imposing on our equations.

Equation (11) becomes

$$y^2 bV + \gamma MW = N, \tag{13}$$

a constant, while equation (12) becomes

$$(y^2b)' - W(y^2b) - \gamma M = 0. \tag{14}$$

In Chandrasekhar's special case, $M = 6m$, $N = 4n(n+1)$. Hence

$$W = \frac{(b/a)' - \gamma}{b/a}. \tag{15}$$

The potential V can be written down in two different ways. First from (13),

$$V = \frac{N - \gamma MW}{y^2 b} = \frac{N}{y^2 b} - \frac{\gamma M}{y^2 b}\left(\frac{(b/a)' - \gamma}{b/a}\right),$$

and second from (7),

$$V = b - W' = b - \left(\frac{(b/a)' - \gamma}{b/a}\right)'. \tag{16}$$

Equating these two values of V, we must have the relation

$$b - \frac{\gamma^2 a^2}{b^2} - \frac{Na}{Mb} - \left(\frac{(b/a)'}{b/a}\right)' = 0, \tag{17}$$

thereby providing a relation between a and b. The remarkable feature about this result is that γ^2 only and not γ enters the equation. Hence this equation is satisfied for the *same* values of a and b whether $\gamma = +1$ or -1, and this gives rise to the dual transformations in every case. This explains why the same right-hand sides occur in Chandrasekhar's equations (53) and (A4), and (54) and (A5) respectively. The change in sign is noted in equations (53) and (A4), and (58) and (A6) respectively, in keeping with the values $\gamma = \pm 1$. The 'surprisingly simple relation' (58) together with (A6) are merely our necessary invariants (13), and fall out as a matter of necessity rather than surprising coincidence.

We may now parametrize the whole class of equations and transformations produced by this analysis. Let $a/b = x$, the only arbitrary functional parameter to be introduced to generate the whole class. Relation (17) now gives

$$b = \gamma^2 x^2 + \frac{Nx}{M} - \left(\frac{x'}{x}\right)'. \tag{18}$$

We may now calculate all the other functions involved in terms of x. The procedure is as follows.

With $x(z)$ a chosen arbitrary function of z, constant parameters M, N and $\gamma(=\pm 1)$ are selected. If required, x may be a function of r, in which case all derivatives with respect to z (denoted by a prime) must be calculated using $\mathrm{d}/\mathrm{d}z = f(r)\,\mathrm{d}/\mathrm{d}r$. Then the auxiliary function b is given by (18), from which it follows that $a = bx$ and $y^2 = M/a = M/bx$. W is then found from (15):

$$W = -\frac{x'}{x} - \gamma x,$$

while (16) and (13) provide two (identical) alternative forms for V:

$$V = b - \left(\frac{(1/x)' - \gamma}{1/x}\right)' = \frac{Nx}{M} - \gamma x W = \frac{Nx}{M} + \gamma x' + \gamma^2 x^2. \tag{19}$$

The parametrized Zerilli equation is given when $\gamma = +1$:

$$Z'' + [k^2 - (NM^{-1}x + x' + x^2)]Z = 0.$$

The parametrized Regge and Wheeler equation is given when $\gamma = -1$:

$$X'' + [k^2 - (NM^{-1}x - x' + x^2)]X = 0.$$

The parametrized Bardeen and Press equation is from (9),

$$\phi'' + \left(k^2 - \frac{2iky'}{y} - \frac{y''}{y} - b\right)\phi = 0.$$

From now onwards, we shall use the symbols V, W, Q, α, β when $\gamma = +1$, and V_0, W_0, Q_0, α_0, β_0 when $\gamma = -1$. The symbols a, b, y, M, N remain unchanged in the two cases. The three identities follow immediately:

$$\alpha - \alpha_0 = 2y(x' - ikx), \qquad \beta - \beta_0 = -2xy, \qquad V - V_0 = 2x'.$$

Usually, there will be transformations between Z and X, as well as between Z, X and ϕ. Using (2), we have

$$\begin{pmatrix}\phi\\ \phi'\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \alpha' - \beta Q & \alpha + \beta'\end{pmatrix}\begin{pmatrix}Z\\ Z'\end{pmatrix},$$

and similarly between ϕ and X. Hence

$$\begin{pmatrix}Z\\ Z'\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \alpha' - \beta Q & \alpha + \beta'\end{pmatrix}^{-1}\begin{pmatrix}\alpha_0 & \beta_0\\ \alpha_0' - \beta_0 Q_0 & \alpha_0 + \beta_0'\end{pmatrix}\begin{pmatrix}X\\ X'\end{pmatrix},$$

where inverse matrices exist provided that the constant in (10) is non-zero.

Apart from a multiplicative constant, we may take the dual transformation to be

$$Z=[(\alpha+\beta')\alpha_0-\beta(\alpha_0'-\beta_0Q_0]X+[(\alpha+\beta')\beta_0-\beta(\alpha_0+\beta_0')]X',$$

where

$$\alpha,\alpha_0=y(Px+x^2-ikx'x^{-1}-2k^2)\pm y(x'-ikx)\equiv y\delta_1\pm y\delta_2,$$

$$\beta,\beta_0=y(-x'x^{-1}+2ik)\pm y(-x)\equiv y\epsilon_1\pm y\epsilon_2,$$

say, from the definitions of α and β and the parametric representations of V and W. The coefficient of X' is more easily found by expressing it as a determinant:

$$\begin{vmatrix}\beta_0 & \alpha_0+\beta_0' \\ \beta & \alpha+\beta'\end{vmatrix}$$

$$=\begin{vmatrix}y\epsilon_1-y\epsilon_2 & y\delta_1-y\delta_2+y'(\epsilon_1-\epsilon_2)+y(\epsilon_1'-\epsilon_2') \\ y\epsilon_1+y\epsilon_2 & y\delta_1+y\delta_2+y'(\epsilon_1+\epsilon_2)+y(\epsilon_1'+\epsilon_2')\end{vmatrix}$$

$$=-2y^2\begin{vmatrix}\epsilon_2 & \delta_2+\epsilon_2' \\ \epsilon_1 & \delta_1+\delta_1'\end{vmatrix} \qquad \text{(upon simplification)}$$

$$=-2y^2[x(-Px-x^2+(x'x^{-1})'] \qquad \text{(upon substitution)}$$

$$=2M,$$

a constant, from the definitions of y^2, a and b.

The coefficient of X is best found by substituting $Z=A(z)X+X'$ into the equation for Z, yielding immediately $A'=x'$, or $A=x+c$. Thus

$$Z=(x+c)X+X'\cdot$$

is the transformation between the equation for Z and its dual, namely between

$$Z''+QZ=0, \qquad X''+(Q+2x')X=0,$$

where $Q=k^2-Px-x'-x^2$. $P=2c$ to fit together correctly. This shows the basic origin of the parametric function x that would yield the Bardeen and Press equation corresponding to the dual pair of equations. We conclude that

$$Z=(x+\tfrac{1}{2}P)X+X'. \tag{20}$$

5. Examples of the transformations

The Chandrasekhar scheme follows by choosing

$$M=6m, \qquad N=4n(n+1), \qquad f(r)=\frac{r-2m}{r}, \qquad x=\frac{3m(r-2m)}{r^2(nr+3m)}.$$

The form for V (when placed on a common denominator) falls out immediately, and all the other associated functions in the scheme are easily derived. Similarly the simpler form for V_0 falls out easily, the sign of x' merely being changed.

As can be seen from the end of the preceding section, any physical wave-propagation equation

$$Z''+[k^2-V(z)]Z=0 \tag{21}$$

has its corresponding dual equation in X via transformation (20), together with its analogous Bardeen and Press equation in ϕ.

When $V(z)$ is given, write

$$V(z) = Px + x' + x^2,$$

with $P = N/M$, $\gamma = 1$. Now let $x = p'/p$ in this general Riccati equation, giving

$$p'' + Pp' - Vp = 0.$$

To reduce this to normal form, write $p = e^{-\frac{1}{2}Pz}s$, giving

$$s'' + (-\tfrac{1}{4}P^2 - V)s = 0, \tag{22}$$

with $x = s's^{-1} - \frac{1}{2}P$.

Consequently, if $Z = Z(z, k, P)$ is a solution of (21), then x can be taken as

$$x = \frac{Z'(z, \frac{1}{2}\mathrm{i}P, P)}{Z(z, \frac{1}{2}\mathrm{i}P, P)} - \tfrac{1}{2}P,$$

though this does not yield necessarily simple associated functions y, W, a, ... The dual equation will be

$$X'' + [k^2 - (V - 2x')]X = 0,$$

since $V - V_0 = 2x'$.

Note that $s = e^{\frac{1}{2}Pz}p = e^{\frac{1}{2}Pz}\exp(\int x\,dz)$. Hence if $x(z)$ is known for equation (20), then $e^{\frac{1}{2}Pz}\exp(\int x\,dz)$ is a solution of (20) when $k^2 = -\frac{1}{4}P^2$, though this does not provide a general solution of (20) for arbitrary k^2.

Thus, in the Chandrasekhar scheme, a solution of the Zerilli equation when $k = \frac{1}{2}\mathrm{i}P = \mathrm{i}n(n+1)/3m$ is given by

$$Z = e^{\frac{1}{2}Pz}\exp\left(\int \frac{3m(r-2m)}{r^2(nr+3m)}\frac{r\,dr}{r-2m}\right) = \frac{r}{nr+3m}e^{n(n+1)z/3m}.$$

To ascertain what happens for the equation of the harmonic oscillator, consider

$$Z'' + (k^2 - z^2)Z = 0.$$

For convenience, write this as

$$Z'' + [k^2 - 2 - (z^2 - 2)]Z = 0,$$

with $V = z^2 - 2$. Equation (22) is

$$s'' + [-\tfrac{1}{4}P^2 - (z^2 - 2)]s = 0,$$

or

$$s'' + (1 - z^2)s = 0$$

when $P = 2$. Then $s = \exp(-\frac{1}{2}z^2)$ for solutions bounded at $z = \pm\infty$, and $x = -z - 1$, so the dual equation becomes

$$X'' + (k^2 - 2 - z^2)X = 0,$$

the transformation being

$$Z \propto zX - X'.$$

This is a well known result for the harmonic oscillator, that if k^2 is an eigenvalue, then so is k^2-2, provided that k^2 is not already the lowest eigenvalue (see Heading 1975, p 300).

Unlike the case of the equations of the black hole, the analogous Bardeen and Press equation is more complicated than the original equation in Z, since poles are included in the new potential. When $\gamma=1$, the associated functions W, b, y^2 have the values

$$W=\frac{z^2+2z}{z+1},\qquad b=\frac{z(z^3+2z^2-2)}{(z+1)^2},\qquad y^2=-\frac{M(z+1)}{z(z^3+2z^2-2)},$$

so a term y'/y in the Bardeen and Press potential will have poles at those points where the factors $z+1$, z, z^3+2z^2-2 vanish.

Similar transformations and results hold for the Schrödinger equation for the hydrogen atom, for which we shall write the radial equation in normal form as

$$Z''+\left(-\kappa^2+\frac{A}{z}-\frac{l(l+1)}{z^2}\right)Z=0,$$

l being a quantum number. Following the derivation of equation (22), write

$$-\frac{A}{z}+\frac{l(l+1)}{z^2}=Px+x'+x^2,$$

or

$$s''+\left(-\frac{1}{4}P^2+\frac{A}{z}-\frac{l(l+1)}{z^2}\right)s=0.$$

A simple solution occurs when $P=A/(l+1)$, when

$$s=\mathrm{e}^{-\frac{1}{2}Pz}z^{l+1},$$

giving

$$x=s's^{-1}-\tfrac{1}{2}P=(l+1-Pz)/z.$$

Consequently the dual equation reduces to

$$X''+\left(-\kappa^2+\frac{A}{z}-\frac{(l+1)(l+2)}{z^2}\right)X=0,$$

the transformation being

$$Z=\left(\frac{l+1}{z}-\frac{A}{2(l+1)}\right)X+X',$$

and the inverse transformation

$$X=\left(\frac{l+1}{z}-\frac{A}{2(l+1)}\right)Z-Z'.$$

Consequently, if κ^2 is an eigenvalue for the quantum number l, it is also for the quantum number $l+1$, provided

$$\kappa^2\neq\frac{A^2}{4(l+1)^2},$$

in which case $Z=s$, being such that X vanishes identically.

A similar calculation may be carried out for the angular momentum of a single particle. In normal form, the relevant equation is

$$Z''+\left(k^2+\frac{1}{4}-\frac{m^2-\frac{1}{4}}{\sin^2\theta}\right)Z=0,$$

the independent variable being θ. Following the above procedure, write

$$-\frac{1}{4}+\frac{m^2-\frac{1}{4}}{\sin^2\theta}=Px+x'+x^2,$$

or

$$s''+\left(-\frac{1}{4}P^2+\frac{1}{4}-\frac{m^2-\frac{1}{4}}{\sin^2\theta}\right)s=0.$$

A special solution exists when $-\frac{1}{4}P^2=m(m+1)$, for which we may take $s=\sin^{m+\frac{1}{2}}\theta$, giving

$$x=(m+\tfrac{1}{2})\cot\theta-\tfrac{1}{2}P.$$

The dual equation then becomes

$$X''+\left(k^2+\frac{1}{4}-\frac{(m+1)^2-\frac{1}{4}}{\sin^2\theta}\right)X=0,$$

possessing the transformation

$$X=(m+\tfrac{1}{2})\cot\theta\, Z-Z'.$$

If k^2 is an eigenvalue for the quantum number m, it also will be for $m+1$, and this process continues until $k^2=m(m+1)$, when $Z=\sin^{m+\frac{1}{2}}\theta$ and $X\equiv 0$.

6. Derivation of the Chandrasekhar scheme from simplifying assumptions

In the three cases just considered, the dual equations for X and Z have been simply derived, but in each case the analogous Bardeen and Press equation in ϕ would be very complicated, owing to y possessing most awkward forms. The question therefore must arise, why should the Bardeen and Press equation in the Chandrasekhar scheme be so relatively simple? In fact, can the simplicity of the Bardeen and Press equation be derived from fundamental assumptions that inevitably lead to the simple form for q? We shall show how this question can be answered in the affirmative.

Throughout this investigation, we shall let a symbol such as m denote the linear form $r-m_1$ when f is not equal to unity, and $z-m_1$ when f equals unity. Forms for x and f are assumed, from which y is calculated, given by

$$y^2=\frac{M}{a}=\frac{M}{bx}=\frac{M}{x^3+Px^2-x(x'/x)'}=\frac{M}{D}, \qquad \text{say.} \tag{23}$$

We assert that the Bardeen and Press potential is 'simple' when: (i) the quantity (23) is a perfect square; and (ii) no further zeros or poles are involved other than those appearing in x and f. These conditions are impossible to fulfil except under certain circumstances.

Firstly, take $x = Al/m$, $f = B$, where l and m are linear factors in r, with A and B constants. Then

$$D=\frac{A^3l^3}{m^3}+\frac{A^2Pl^2}{m^2}+\frac{AB^2}{lm}-\frac{AB^2l}{m^3}.$$

The third term contains a single l in its denominator. It cannot cancel with anything in the numerator, so prevents D from being a perfect square.

Secondly, take $x = Al/m$, $f = Bn/p$, with $l \neq m$, $n \neq p$. Then

$$D=\frac{A^3l^3}{m^3}+\frac{A^2Pl^2}{m^2}-\frac{AB^2n}{mp^2}+\frac{AB^2n^2}{mp^3}+\frac{AB^2n^2}{mp^2l}+\frac{AB^2ln}{m^2p^2}-\frac{AB^2ln^2}{m^2p^3}-\frac{AB^2ln^2}{m^3p^2}.$$

The fifth term, alone with l in its denominator, prevents D from being a perfect square, unless cancellation is possible. We must choose $n \equiv l$, and then the third and fifth terms cancel. Again, the fourth and seventh terms involving p^3 are

$$\frac{AB^2l^2(m-l)}{m^2p^3}.$$

The excess factor p that prevents D being a perfect square cannot be cancelled out, since $m-l$ is a constant, not containing r. No other simplification is possible, so these forms for x and f are not satisfactory to achieve our object.

Thirdly, take $x = Al/m^2$, $f = Bn/p$, with $l \neq m$, $n \neq p$. As before, one single l occurs in the denominator of one term in D; this necessitates $n \equiv l$, so we shall assume this result without further discussion. Then

$$D=\frac{A^3l^3}{m^6}+\frac{A^2Pl^2}{m^4}+\frac{AB^2l^2}{m^2p^3}+\frac{2AB^2l^2}{m^3p^2}-\frac{2AB^2l^3}{m^3p^3}-\frac{2AB^2l^3}{m^4p^2}.$$

The third and fifth terms involving $1/p^3$ now prevent a perfect square. These terms are

$$\frac{AB^2l^2(m-2l)}{m^3p^3}, \tag{24}$$

so p must divide $m-2l$. Hence

$$(r-m_1)-2(r-l_1)\propto r-p_1,$$

leading to

$$p_1 \equiv 2l_1 - m_1 \tag{25}$$

and

$$\frac{m-2l}{p}=-1.$$

Thus term (24) equals $-AB^2l^2/m^3p^2$. This, together with the fourth and sixth terms in D, now gives

$$D=\frac{Al^2}{m^6}\left(A^2l+APm^2-\frac{B^2m^2}{p}\right).$$

The single p in the denominator prevents D from being a perfect square, unless $p = m$; (25) then implies $m = l$, which is contrary to our original requirements. This means that the forms for x and f are unsatisfactory.

Finally, consider the choice $x = Al/m^2p, f = Bl/q$, the l in the numerator of f being chosen for the same reason as before. No cancellations are allowed in x and f. Then

$$D = \frac{A^3l^3}{m^6p^3} + \frac{A^2Pl^2}{m^4p^2} + \frac{AB^2l^2}{m^2pq^3} + \frac{2AB^2l^2}{m^3pq^2}$$
$$- \frac{2AB^2l^3}{m^3pq^3} + \frac{AB^2l^2}{m^2p^2q^2} - \frac{AB^2l^3}{m^2p^2q^3} - \frac{2AB^2l^3}{m^4pq^2} - \frac{AB^2l^3}{m^2p^3q^2}.$$

The third, fifth and seventh terms involving $1/q^3$ prevent D from being a perfect square, unless

(i) q divides $AB^2l^2(mp - 2lp - lm)$;
(ii) $q = m$;
(iii) $q = p$.

Case (i) we shall not investigate further. For a perfect square, case (iii) would imply $q = l$ from terms 7 and 9, not allowed by our initial conditions. Case (ii) gives

$$D = \frac{Al^2}{m^6}\left(\frac{A^2l - B^2lm^2}{p^3} + \frac{APm^2 + B^2m^2 - B^2lm}{p^2} + \frac{3B^2m - 4B^2l}{p}\right).$$

To simplify this, write $L_1 = l_1 - p_1$, so $l = p - L_1$, and the same for m. In partial fractions,

$$D = \frac{Al^2}{m^6}\left(\frac{L_1(B^2M_1^2 - A^2)}{p^3} + \frac{A^2 + APM_1^2 - 3B^2L_1M_1}{p^2}\right.$$
$$\left. + \frac{6B^2L_1 - 2B^2M_1 - 2APM_1}{p} - 3B^2 + B^2M_1^2 + APM_1^2\right).$$

To eliminate an overall odd power of p in the denominator, the coefficient of $1/p^3$ must vanish. Hence choose $A = BM_1$ (the other sign would merely change the sign of P), giving

$$D = \frac{B^2M_1l^2}{m^6}\left(\frac{M_1(BM_1 + M_1^2P - 3BL_1)}{p^2} - \frac{2(BM_1 + M_1^2P - 3BL_1)}{p}\right.$$
$$\left. + (-3B + BM_1^2 + PM_1^3)\right).$$

To be a perfect square, yet not introducing further factors in the numerator, choose

$$BM_1 + M_1^2P - 3BL_1 = 0, \tag{26}$$

enabling us to write D as

$$D = \frac{B^2M_1l^2 \,.\, 3B(L_1M_1 - 1)}{m^6}.$$

y is therefore proportional to m^3/l.

Overall, the conditions for simplicity are

$$m_1 = q_1,$$
$$A = BM_1 = B(m_1 - p_1),$$

and from (26),

$$P=\frac{B(3L_1-M_1)}{M_1^2}=\frac{B(3l_1-m_1-2p_1)}{(m_1-p_1)^2}.$$

In Chandrasekhar's scheme, the values of all numerical parameters are

$$A=3m/n, \qquad B=1, \qquad l_1=2m,$$

$$m_1=0, \qquad p_1=-3m/n, \qquad q_1=0, \qquad P=2n(n+1)/3m.$$

It is obvious that these satisfy the conditions. This investigation also explains the 'mysterious' factor $(nr+3m)\equiv n(r-p_1)$ that appears throughout the theory, though not in y. Based on our simplification procedure, its appearance was inevitable, given A, B and m_1, since the equation $A=BM_1$ gives $p_1=m_1-A/B$.

References

Bardeen J M and Press W H 1973 *J. Math. Phys.* **14** 7–19
Chandrasekhar S 1975 *Proc. R. Soc.* A **343** 289–98
Heading J 1975 *Ordinary Differential Equations, Theory and Practice* (London: Elek Science)
—— 1977 to be published
Regge T and Wheeler J A 1957 *Phys. Rev.* **108** 1063–9
Zerilli F J 1970 *Phys. Rev.* D **2** 2141–60